

ALL-ORDERS FINITE $N = 1$ SUPER-YANG-MILLS THEORIES¹

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Abstract

I present a criterion for all-order finiteness in $N = 1$ SYM theories. The structure of the super-current anomaly, the Callan-Symanzik equation and the supersymmetric non-renormalization theorem for chiral anomalies are the essential ingredients of the proof.

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1 Introduction

I shall report in this talk about a criterion for all-order finiteness in $N = 1$ supersymmetric grand unified theories (SGUTs). By all-order finiteness, I mean the vanishing of all β -functions, both gauge and Yukawa, at every order of perturbation theory. This finiteness criterion [1] has been on the market for some time, and it is now being applied to phenomenological models. Following a talk in that direction by G. Zoupanos [2], I would like to present in some details the criterion itself, as well as some steps of its derivation.

Before I start, let me mention that there exist related (and slightly different) approaches to all-order finiteness in $N = 1$ SYM [3, 4]. Due to lack of time, I shall not comment on these.

2 $N = 1$ Super-Yang-Mills Theory

We consider an $N = 1$ SYM theory with a simple gauge group G . The real gauge and chiral matter superfields are resp. denoted by ϕ and¹ A^R . The gauge-invariant superfield action writes (for conventions, see [1, 5]):

$$S^{\text{inv.}} = - \frac{1}{128g^2} \text{Tr} \int d^4x \, DD \, F^\alpha F_\alpha + \frac{1}{16} \int d^4x \, DD \, \bar{D}\bar{D} \sum_R \bar{A}_R e^{\phi_i T_R^i} A^R + \frac{1}{6} \left(\int d^4x \, DD \, \lambda_{rst} A^r A^s A^t + \text{c.c.} \right). \quad (2.1)$$

The gauge-fixing of the above action [5] is out of our purpose. Following the BRS quantization procedure, we construct the vertex functional as $\Gamma = S^{\text{inv.}} + S^{\text{gauge fixing}} + S^{\text{Faddeev-Popov}} +$ loop corrections of order \hbar^n , and define the quantum theory generating functional to be the most general solution of a set of constraints given by the gauge condition, the equations of motion, the rigid and BRS symmetries, etc. A subset of these constraints is relevant to this presentation:

1. \mathcal{R} -symmetry. Infinitesimally $\delta_{\mathcal{R}}\varphi = i(n_\varphi + \theta^\alpha \partial_{\theta^\alpha} - \bar{\theta}^{\dot{\alpha}} \partial_{\bar{\theta}^{\dot{\alpha}}})\varphi$ on a generic superfield $\varphi = A, \bar{A}, \phi, c_+, \bar{c}_+, \dots$, with \mathcal{R} -weights $n_A = -n_{\bar{A}} = -2/3$, $n_{\varphi=\phi, c_+, \bar{c}_+} = 0$. The functional \mathcal{R} -Ward identity writes $\mathcal{W}_{\mathcal{R}}\Gamma = 0$.

2. Supersymmetry, expressed through the Ward identities $\mathcal{W}_\alpha\Gamma = 0$, $\bar{\mathcal{W}}_{\dot{\alpha}}\Gamma = 0$.

3. BRS invariance, acting infinitesimally as $s e^\phi = e^\phi c_+ - \bar{c}_+ e^\phi$, $s A^r = -c_{+i}(T_R^i)^\rho A^{(R,\sigma)}$, and $s c_+ = -\frac{1}{2}\{c_+, c_+\}$. BRS invariance is encoded in a (non-linear) Ward identity, the Slavnov identity $\mathcal{S}(\Gamma) = 0$, which is satisfied provided there is no gauge anomaly [5, 6].

4. A possible set of rigid chiral symmetries: $\delta_a A^R = i e_a^R S A^S$, $\delta_a \bar{A}_R = -i \bar{A}_S e_a^S$, gener-

¹For matter superfields, we use a compact indices convention: R denotes both the field and its representation. We also define $r \equiv (R, \rho)$, where ρ denotes the field components within a given representation R .

ated by Hermitean charges $e_a = e_a^\dagger$ ($\delta_a \phi = \delta_a c_+ = 0$). The “chiral” Ward identity² $\mathcal{W}_a \Gamma = 0$ is satisfied provided $\lambda_{rsu} e_a^u + \text{cyclic permutations}(r, s, t) = 0$.

3 Supercurrent and Anomalies

The Ward operators for supersymmetry, translations and \mathcal{R} -invariance obey a superPoincaré algebra. As a consequence, there exists a superfield Ward operator $\hat{\mathcal{W}} = \mathcal{W}_R - i \theta^\alpha \mathcal{W}_\alpha + i \bar{\theta}^{\dot{\alpha}} \bar{\mathcal{W}}_{\dot{\alpha}} - 2(\theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}}) \mathcal{W}_\mu^T + \dots$. The component Noether currents associated to \mathcal{R} -symmetry, supersymmetry and translation invariance form the supercurrent [7, 8] $V_\mu(x, \theta, \bar{\theta}) = R_\mu(x) - i \theta^\alpha Q_{\mu\alpha}(x) + i \bar{\theta}^{\dot{\alpha}} \bar{Q}_{\mu\dot{\alpha}}(x) - 2(\theta^\alpha \sigma_{\alpha\dot{\alpha}}^\nu \bar{\theta}^{\dot{\alpha}}) T_{\mu\nu}(x) + \dots$, which satisfies the supertrace equation:

$$\hat{\omega} \Gamma = \partial^\mu V_\mu + i (DD \mathbf{S} - \bar{D}\bar{D} \bar{\mathbf{S}}) . \quad (3.1)$$

The chiral superfield \mathbf{S} in the right side is the supercurrent anomaly. In components, it yields an Abelian chiral anomaly which breaks the R -current divergence:

$$\partial_\mu R^\mu = i \hat{\omega} \Gamma|_{\theta=0} + i (DD \mathbf{S} - \bar{D}\bar{D} \bar{\mathbf{S}}) , \quad (3.2)$$

as well as dilatation anomalies in the energy-momentum “trace”:

$$\int d^4x T^\mu_\mu = \mathcal{W}^D \Gamma - \frac{1}{3} \int d^4x (DD \mathbf{S} + \bar{D}\bar{D} \bar{\mathbf{S}}) . \quad (3.3)$$

Here \mathcal{W}^D is the Ward operator of dilatations: $\delta_D \varphi = (d_\varphi + x^\mu \partial_\mu + \frac{1}{2} \theta^\alpha \partial_{\theta^\alpha} + \frac{1}{2} \bar{\theta}^{\dot{\alpha}} \partial_{\bar{\theta}^{\dot{\alpha}}}) \varphi$.

Our task is to relate the Abelian chiral anomaly in the \mathcal{R} -current divergence and the dilatation anomalies in the energy-momentum “trace” to the Abelian anomalies associated to the (possible) chiral symmetries \mathcal{W}_a . The natural setting for deriving such a relation is provided by the Callan-Symanzik equation. We shall arrive at its formulation by expanding the supercurrent anomaly \mathbf{S} in a basis of dimension 3, BRS-invariant, chiral insertions $\{L_i\}$ as:

$$\mathbf{S} = \beta_g L_g + \sum_{\lambda_{rst}} \beta_{rst} L_{rst} - \sum_{R,S} \gamma_R^S L_S^R + \dots , \quad (3.4)$$

where the dots stand for insertions which are not essential in the present context, and the L_i ’s are defined by:

$$\begin{aligned} \int d^4x (DD L_g + \bar{D}\bar{D} \bar{L}_g) &\equiv \partial_g \Gamma \\ \int d^4x (DD L_{rst} + \bar{D}\bar{D} \bar{L}_{rst}) &\equiv \partial_{\lambda_{rst}} \Gamma \\ \int d^4x (DD L_S^R + \bar{D}\bar{D} \bar{L}_S^R) &\equiv \mathcal{N}_S^R \Gamma = \int d^4x \left(DD A^R \frac{\delta}{\delta A^S} + \bar{D}\bar{D} \bar{A}_S \frac{\delta}{\delta \bar{A}_R} \right) \Gamma . \end{aligned} \quad (3.5)$$

Inserting into the energy-momentum “trace” the expansion for \mathbf{S} and the forms of the L_i ’s, and relating the (broken) Ward identity of dilatations to the scaling operator through the dimensional analysis identity $\mathcal{W}^D \Gamma = \sum_{\mu_i} \mu_i \partial_{\mu_i}$, one arrives at:

$$C\Gamma \equiv \left[\sum_{\mu_i} \mu_i \partial_{\mu_i} + \beta_g \partial_g + \sum_{\lambda_{rst}} \beta_{rst} \partial_{\lambda_{rst}} - \sum_{R,S} \gamma_R^S \mathcal{N}_S^R + \dots \right] \Gamma = 0 . \quad (3.6)$$

²The Ward identities for \mathcal{R} -symmetry, supersymmetry, BRS symmetry, as well as the chiral one, are taken to hold up to soft, supersymmetric mass terms.

This is the Callan-Symazik equation, which describes how dilation invariance is broken by the β -functions β_g , β_{rst} associated to the renormalization of the gauge, resp. Yukawa couplings, and by the anomalous dimensions γ_R^S .

Let us now perform a change of basis for the counting operators \mathcal{N}_S^R : $\{\mathcal{N}_S^R\} \rightarrow \{\mathcal{N}_{0a} \equiv e_{0a}^R{}_S \mathcal{N}_S^R\} \oplus \{\mathcal{N}_{1k}\}$, where the $e_{0a}^R{}_S$ are charge matrices corresponding to the center of the algebra of chiral symmetries $\{\mathcal{W}_a\}$ (*i.e.*, $[\mathcal{W}_{0a}; \mathcal{W}_b] = 0, \forall b$), and the new counting operators annihilate the superpotential: $\mathcal{N}_{0a}(\int d^4x \ DD \lambda_{rst} A^r A^s A^t) = 0$.

Next, one can show that the supercurrent anomaly \mathbf{S} , as well as each of the insertions L_i of its expansion in the new basis (omitting the unessential term L_{1k}), can be written as:

$$\begin{aligned} \mathbf{S} &= \bar{D}\bar{D} (r K_3^0 + \dots) , \\ L_g &= \bar{D}\bar{D} \left(\frac{1}{128g^3} + r_g \right) K_3^0 + \dots , \\ L_{rst} &= \bar{D}\bar{D} r_{rst} K_3^0 + \dots , \\ L_{0a} &= \bar{D}\bar{D} r_{0a} K_3^0 + \dots , \end{aligned} \tag{3.7}$$

where the dots stand for invariant currents and “genuinely chiral” terms which cannot be written as $\bar{D}\bar{D}(\dots)$. Replacing these expressions into $\mathbf{S} = \beta_g L_g + \sum_{\lambda_{rst}} \beta_{rst} L_{rst} - \sum_a \gamma_{0a} L_{0a} + \dots$, and identifying the coefficients of the K_3^0 -dependent terms, yields the relation:

$$r = \beta_g \left(\frac{1}{128g^3} + r_g \right) + \sum_{\lambda_{rst}} \beta_{rst} r_{rst} - \sum_a \gamma_{0a} r_{0a} . \tag{3.8}$$

4 Non-renormalization of Chiral Anomalies

Specializing to the case under consideration, the non-renormalization theorem for chiral anomalies in $N = 1$ SYM (see [1, 9]) tells us that r and r_{0a} in (3.8) are non-renormalized, *i.e.*, they are strictly of order \hbar .

r is the coefficient of the Abelian anomaly in the \mathcal{R} -axial current, and the r_{0a} ’s are the coefficients of the Abelian anomalies of the axial currents associated to the chiral \mathcal{W}_a -symmetries. r and r_{0a} are given by their one-loop values [10, 9, 1]:

$$r = \frac{1}{128g^3} \beta_g^{(1)} = \frac{1}{512(4\pi)^2} \left(\sum_R T(R) - 3C_2(G) \right) , \quad r_{0a} = -\frac{1}{256(4\pi)^2} \sum_R e_a^R{}_R T(R) . \tag{4.1}$$

Note: The proof of the non-renormalization theorem uses the fact that the three-form K_3^0 , the supersymmetric Chern-Simons form, is related through the supersymmetric descent equations to the zero-form $K_0^3 = \frac{1}{3} \text{Tr } c_+^3$, the cubed ghost field insertion. The non-renormalization theorem for chiral vertices guarantees the finiteness of the latter insertion.

5 Criterion for all-order vanishing β -functions

Criterion: Consider an $N=1$ super-Yang-Mills theory with simple gauge group. If

- (i) there is no gauge anomaly,
- (ii) the gauge β -function vanishes at one loop:

$$\beta_g^{(1)} = 0 , \quad (5.1)$$

- (iii) there exist solutions of the form $\lambda_{rst} = \lambda_{rst}(g)$ to the conditions of vanishing one-loop anomalous dimensions

$$\gamma^{(1) R}_S = 0 , \quad (5.2)$$

and (iv) this solution is *isolated* and *non-degenerate* when considered as a solution of the conditions of vanishing one-loop Yukawa β -functions:

$$\beta_{rst}^{(1)} = \lambda_{rsu} \gamma^{(1) u}_t + \text{cyclic permutations}(r, s, t) = 0 , \quad (5.3)$$

then the theory depends on a single coupling constant (the gauge coupling g) with a β -function which vanishes at all orders.

Let us give a sketch of the proof. With the expressions for r and r_{0a} (4.1), it follows from (ii) and (iii) that $r = 0$, resp.³ $r_{0a} = 0$. Then (3.8) reduces to

$$0 = \beta_g \left(\frac{1}{128 g^3} + r_g \right) + \sum_{\lambda_{rst}} \beta_{rst} r_{rst} . \quad (5.4)$$

That the Yukawa couplings λ_{rst} are proportional to g in the one-loop approximation as a consequence of (iii) is clear from [10]:

$$\gamma^{(1) r}_s = \frac{1}{(2\pi)^2} \left(\bar{\lambda}^{ruv} \lambda_{suv} - \frac{1}{16} g^2 C_2(R) \delta_s^r \right) . \quad (5.5)$$

At higher orders, $\lambda_{rst} = \lambda_{rst}(g)$ are formal power series in g , and one needs to impose for consistency that these functions satisfy the reduction equations [11]:

$$\beta_{rst} = \beta_g \frac{\partial \lambda_{rst}}{\partial g} . \quad (5.6)$$

One can show [1] that a solution to these equations exists at all orders (and is unique) if the lowest-order solution is isolated and non-degenerate. At one-loop, eq. (5.6) reduces to $\beta_{rst}^{(1)} = 0$; this is just hypothesis (iv).

Next one replaces (5.6) into (5.4) and gets:

$$0 = \beta_g \left(\frac{1}{128 g^3} + r_g + \sum_{\lambda_{rst}} \frac{\partial \lambda_{rst}}{\partial g} \right) . \quad (5.7)$$

The parenthesis being perturbatively invertible, it follows that $\beta_g = 0$ at all orders, for the unique remaining (independent) coupling of the theory, *e.g.*, the gauge coupling g .

Note that the above criterion guarantees finiteness of the theory at all orders, although its conditions involve exclusively one-loop quantities. The conditions $\beta_g^{(1)} = \gamma^{(1) R}_S = 0$ are known

³One uses here a corollary to the main non-renormalization theorem stated in Section 4: the conditions $\gamma^{(1) R}_S = 0$ are compatible *iff* $r_{0a} = 0$.

to guarantee one- and two-loop vanishing of the β -functions [10]. Models which fulfill these conditions are tabulated *e.g.*, in [12], for the most popular (simple) gauge groups. Conditions (iii) and (iv) represent therefore consistency requirements that are necessary in order to extend the vanishing of the β -functions at all orders. To ensure the unicity and non-degeneracy of the solution of $\gamma^{(1) R}_S = 0$ *considered as a solution of* $\beta_{rst}^{(1)} = 0$, one is led to constrain the model by imposing additional, chiral symmetries. One expects that such additional symmetries, for some relevant gauge group, should turn out to have physical significance and predictive power.

Some models satisfying the all-order finiteness criterion are known. An $SU(6)$ SYM theory has been presented in [1]. Other attempts at finding all-order finite models have resulted in constraining the initial theory by imposing orbifold symmetries [2].

Let me mention that the criterion above can neither be used for Abelian gauge theories, nor for semi-simple gauge groups containing $U(1)$ factors. This is a direct consequence of the form of $\beta_g^{(1)}$ (see (4.1)). The $U(1)$ quadratic Casimir being zero, the corresponding $\beta_{g[U(1)]}^{(1)} \neq 0$ and the condition (ii) of the criterion cannot be satisfied. This is however physically fine since one expects a low-energy theory with a $U(1)$ factor in its gauge group to be an effective theory, and all-order finiteness to be realized only above the unification scale.

6 Conclusions

In this talk, I have presented a criterion for all-order vanishing β -functions, *i.e.*, perturbative finiteness, in $N = 1$ super-Yang-Mills theories. Finiteness is expected only at the grand unified level, since low-energy, effective theories are expected to contain a $U(1)$ factor in their gauge group.

A systematic search for finite SGUTs is made possible by the fact that the hypotheses of the criterion involve one-loop quantities only. Examples of finite $N = 1$ SGUTs exist, but no complete classification has been achieved to date. *The process of testing for all-order finiteness of a given model is constructive in the sense that it yields the global symmetries of the superpotential.* Indeed, one has to look for a unique and non-degenerate solution of the form $\lambda_{\text{Yukawa}} = \lambda_{\text{Yukawa}}(g)$ to the conditions of vanishing one-loop Yukawa β -functions. Such a solution does generally not exist, and requires that one restricts the superpotential until uniqueness and non-degeneracy are attained.

Finite SGUTs with model-dependent global (Lie group) or discrete symmetries should provide an interesting setting for phenomenology. Applying the criterion presented here could reveal a precious guide to family symmetry, or to orbifold-type discrete symmetries resulting from compactification, as well as to the symmetries of interest for astrophysics, to mention only a few.

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